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$T=0$ Pion-Pion Scattering

J. J. BREHM*
Department of Physics, Northwestern University, Evanston, Illinois
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The several-channel N-over-D method for pion-pion scattering in the variational framework proposed by Kreps *et al.* is modified to provide for a better controlled procedure of parameter variation. The choice of channels is such that the ghost constraints can be imposed for isospin $T=0$ without introducing any parameters other than those specifying the coupling between channels. The crossing relations yield a condition on the $T=0$ and $T=2$ full amplitudes which can then be studied by means of a thorough computer search among the parameters. The s-wave ghost is found to be at $s=-40$. The s-wave scattering lengths are computed and the s- and d-wave phase shifts are determined. As in the earlier model, the s waves are found to be repulsive and the $T=0$ d wave shows no f^0 resonance. Another choice of trial functions is discussed and illustrated by means of a model of scalar meson scattering.

1. INTRODUCTION

RECENTLY, a method departing from the frequently adopted bootstrap philosophy has been presented for a unitary calculation of pion-pion scattering.[1] Reference has already been made in I to other approaches to the problem so these will not be cited again here. The central part of the method described in I is a variational procedure performed to optimize the fit to the crossing relations; the content of this paper is a modification of that method providing for a much more easily controlled variation of the parameters of the model.

As in I, the many-channel N-over-D method is used for the construction of unitary amplitudes. Here a different choice of inelastic channels is made; it is the selection of these channels, in particular the selected number of them, which brings the variation of parameters in the isospin $T=0$ and $T=2$ channels under control. This will be described in Sec. 2 along with a brief reiteration of the techniques introduced in I. After the parameters have been determined to optimize the fit to the crossing relations within the framework of this model, the phase shifts can then be calculated. In Sec. 3 the s- and d-wave phase shifts for $T=0$ and $T=2$ are given along with the s-wave scattering lengths. In Sec. 4 a discussion is given of possible other trial functions which might be useful in this model.

2. METHOD

To best illustrate wherein the proposed improvement on the model of I lies, it is appropriate to sketch briefly the methods described in I.

The channels chosen to couple via unitarity to the $(\pi\pi)$ channel (channel 1) were taken to be $(\rho\rho)$ in $T=0$ and $T=2$, and $(\pi\omega)$ in $T=1$. The partial wave amplitudes were written in matrix form as:

$$M_l^{(T)}=h^{(T)l/2}N_l^{(T)}D_l^{(T)-1}h^{(T)l/2} \tag{1}$$

in which the elements of the diagonal matrix $h^{(T)}$ were the Khuri threshold factors appropriate to the various channels. The elements of the matrix N_l were taken to be l-independent and represented the trial functions for a variational calculation to satisfy the crossing relations for pion-pion scattering:

$$\begin{aligned}M_{11}^{(0)}(s,t,u)&=\tfrac{1}{3}M_{11}^{(0)}(t,s,u)+M_{11}^{(1)}(t,s,u)+(5/3)M_{11}^{(2)}(t,s,u)\,,\\ M_{11}^{(1)}(s,t,u)&=\tfrac{1}{3}M_{11}^{(0)}(t,s,u)+\tfrac{1}{2}M_{11}^{(1)}(t,s,u)-\tfrac{5}{6}M_{11}^{(2)}(t,s,u)\,,\\ M_{11}^{(2)}(s,t,u)&=\tfrac{1}{3}M_{11}^{(0)}(t,s,u)-\tfrac{1}{2}M_{11}^{(1)}(t,s,u)+\tfrac{1}{6}M_{11}^{(2)}(t,s,u)\,.\end{aligned} \tag{2}$$

Born approximation suggested suitable trial functions; the procedure then reduced to varying all the multiplicative parameters in N. The ghost constraints were applied to reduce the number of parameters: the requirements were imposed that

$$\mathfrak{D}_{l=1}^{(T=0)}(s=0)=0\,, \tag{3}$$

corresponding to the Pomeranchukon on the vacuum trajectory,[2,3] where

$$\mathfrak{D}_l^{(T)}=\det D_l^{(T)}\,, \tag{4}$$

and

$$\mathfrak{D}_{l=0}^{(T=0)}(s=s_g)=0\,, \tag{5}$$

corresponding to the ghost on the vacuum trajectory, and finally

$$\mathfrak{N}_{l=0}^{(T=0)}(s=s_g)=0\,, \tag{6}$$

corresponding to the vanishing of the residue at the ghost in each element of the matrix $\mathfrak{N}_0^{(0)}$ defined by

$$N^{(T)}D_l^{(T)-1}=\mathfrak{N}_l^{(T)}/\mathfrak{D}_l^{(T)}\,. \tag{7}$$

The value of s_g was determined in the search. Imposition of the constraints (3), (5), and (6) called for the introduction of additional parameters[4] for which one had

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[1] R. E. Kreps, L. F. Cook, J. J. Brehm, and R. Blankenbecler, Phys. Rev. 133, B1526 (1964). This work will be referred to as I in the subsequent text.

[2] I. I. Pomeranchuk, Zh. Eksperim. i Teor. Fiz. 34, 725 (1958) [English transl.: Soviet Phys.—JETP 7, 499 (1958)].

[3] G. F. Chew and S. C. Frautschi, Phys. Rev. Letters 8, 41 (1962).

[4] These parameters were denoted in I by $s_0^{(T)}$ and $s_1^{(T)}$, and were incorporated for generality into all three isospin channels.

very little feeling as to their value and for which physical significance was lacking.

It is the purpose of this paper to apply the philosophy of I introducing only the multiplicative parameters of the N matrix which are in a sense physical since they measure the coupling between channels; furthermore, it is imperative that the number of parameters treated be sufficiently small as to allow a thorough search producing the best possible fit within the model.

Let the choice of inelastic channels be as follows: in isospin $T=1$ use $(\pi\omega)$ states (channel 2), and $(K\bar{K})$ states (channel 3); in $T=0$ use $(K\bar{K})$ states and also $(\eta\eta)$ states (channel 4); in $T=2$ use no inelastic states. The selection of three $T=0$ channels precludes the need for additional parameters. The number of independent $T=0$ parameters after imposing the ghost constraints (3), (5), and (6) is two and, with only a single $T=2$ parameter, allows a very thorough search in fitting the relation[5]

$$\Delta^{(0)}=-2\Delta^{(2)}. \tag{8}$$

For a study of low and medium energy pion-pion scattering this choice of states would seem more appropriate than the selection made in I. The fact that $T=2$ scattering is taken here to be a one-channel problem was suggested in the analysis of I where it was found that $(\pi\pi)$ and $(\rho\rho)$ preferred to uncouple. This choice of channels also does not call for any model of the inelastic spin dependence of the sort developed in I for the $(\rho\rho)$ states.

The Khuri threshold matrices in Eq. (1) are

$$h^{(0)}=\begin{bmatrix} h_1 & 0 & 0\\ 0 & h_3 & 0\\ 0 & 0 & h_4\end{bmatrix},\quad h^{(1)}=\begin{bmatrix} h_1 & 0 & 0\\ 0 & h_2 & 0\\ 0 & 0 & h_3\end{bmatrix},\quad h^{(2)}=(h_1), \tag{9}$$

in which, for each i, the h_i's are related to the c.m. momenta p_i by

$$p_i^2=4h_i/(1-h_i)^2. \tag{10}$$

Equation (1) therefore displays the correct threshold behavior of the partial waves and, as pointed out by Khuri,[6] the asymptotic l behavior given in (1) yields the correct two-particle thresholds of the summed amplitudes in the crossed channels. That this is true for the diagonal elements of the amplitude (and only these) can be seen by looking at first Born approximation: if $D_{l,ij}{}^{(T)}=\delta_{ij}$ then the diagonal summed amplitudes become, under the assumption that $N^{(T)}$ is l-independent:

$$M_{ii\mathrm{Born}}{}^{(T)}=\sum_l (2l+1)(h_{ii}{}^{(T)})^l N_{ii}{}^{(T)}P_l(\cos\theta). \tag{11}$$

For the identical particle channels the sum is over even l for $T=0$ and $T=2$, and over odd l for $T=1$. The results are

$$M_{11\mathrm{Born}}{}^{(T)}=4N_{11}\frac{1+h_1}{(1-h_1)^2}\left[\frac{1}{(4-t)^{3/2}}\pm\frac{1}{(4-u)^{3/2}}\right],\quad T=\begin{Bmatrix}\text{even}\\ \text{odd}\end{Bmatrix},$$

$$M_{jj\mathrm{Born}}{}^{(T)}=8N_{jj}\frac{1+h_j}{(1-h_j)^2}\frac{1}{(4-t)^{3/2}},\quad j=2,3 \tag{12}$$

$$M_{44\mathrm{Born}}{}^{(T)}=4N_{44}\frac{1+h_4}{(1-h_4)^2}\left[\frac{1}{(4-t)^{3/2}}+\frac{1}{(4-u)^{3/2}}\right],\quad T=0.$$

As pointed out in I, the use of the Khuri factors also leads to the correct thresholds of the respective imaginary parts in second Born approximation; i.e., the imaginary parts are singular on the relevant Mandelstam spectral curves.

Equations (12) suggest the trial functions to be adopted in this model:

$$N^{(T)}=F^{(T)}a^{(T)} \tag{13}$$

in which the $F^{(T)}$'s are diagonal matrices with elements

$$F_{ii}{}^{(T)}=(1-h_i)^2/(1+h_i), \tag{14}$$

with $i=1, 3, 4$ for $T=0$, $i=1, 2, 3$ for $T=1$, and $i=1$ for $T=2$. The matrices $a^{(T)}$ contain the parameters to be varied such that the fit to Eqs. (2) by the partial wave sums is optimized:

$$a^{(0)}=\begin{bmatrix} a_1{}^{(0)} & c^{(0)} & d^{(0)}\\ c^{(0)} & a_3{}^{(0)} & e^{(0)}\\ d^{(0)} & e^{(0)} & 0\end{bmatrix},\quad a^{(1)}=\begin{bmatrix} a_1{}^{(1)} & b^{(1)} & c^{(1)}\\ b^{(1)} & 0 & f^{(1)}\\ c^{(1)} & f^{(1)} & a_3{}^{(1)}\end{bmatrix},\quad a^{(2)}=(a_1{}^{(2)}). \tag{15}$$

Unitarity requires that

$$\mathrm{Im}D_l{}^{(T)}=-\rho h^{(T)l}N^{(T)}, \tag{16}$$

where ρ is the usual diagonal matrix of phase-space factors ρ_i written as[7]

$$\rho_i=p_i/(2s^{1/2})=h_i^{1/2}/[s^{1/2}(1-h_i)]. \tag{17}$$

[5] As in I the Δ's are defined by:

$$\Delta^{(T)}(s,t)=M_{11}{}^{(T)}(s,t)-M_{11}{}^{(T)}(t,s).$$

[6] N. N. Khuri, Phys. Rev. **130**, 429 (1963).

[7] As pointed out in I, factors of 2π, etc., in the phase-space factors are irrelevant and can be regarded as absorbed into the parameter matrix $a^{(T)}$.

The elements of $D_l^{(T)}$ then are

$$D_{l,ij}^{(T)}=\delta_{ij}-H_i^{(l)}(s)a_{ij}^{(T)}, \tag{18}$$

where

$$H_i^{(l)}(s)=\frac{1}{\pi}\int_{s_i}^{\infty}\frac{ds'}{s'-s}\frac{h_i'^{l+\frac{1}{2}}}{s'^{1/2}}\frac{1-h_i'}{1+h_i'}. \tag{19}$$

The problem of determining $T=0$ and $T=2$ pion-pion scattering is carried out now by varying parameters to fit Eq. (8). This would involve a search over six parameters, as can be seen from expression (15). The ghost constraints reduce the number to three independent parameters for which a computer search can be readily carried out.

The ghost constraints in the $T=0$ channel are imposed as follows. The Pomeranchukon at $s=0$ for $l=1$ calls for:

$$\begin{aligned}\mathfrak{D}_1^{(0)}(0)=&[1-P_1(a_1^{(0)}+d^{(0)2}P_4]\\&\times[1-P_3(a_3^{(0)}+e^{(0)2}P_4)]\\&-P_1P_3(c^{(0)}+d^{(0)}e^{(0)}P_4)^2\\=&0\end{aligned} \tag{20}$$

in which the P_i's are defined to be:

$$P_i=H_i^{(l=1)}(0). \tag{21}$$

The ghost at $s=s_g$ for $l=0$ calls for

$$\begin{aligned}\mathfrak{D}_0^{(0)}(s_g)=&[1-G_1(a_1^{(0)}+d^{(0)2}G_4)]\\&\times[1-G_3(a_3^{(0)}+e^{(0)2}G_4)]\\&-G_1G_3(c^{(0)}+d^{(0)}e^{(0)}G_4)^2\\=&0\end{aligned} \tag{22}$$

in which

$$G_i=H_i^{(l=0)}(s_g). \tag{23}$$

In order that the residue of $M_{0,11}^{(0)}$ vanish at $s=s_g$ it is necessary that

$$\begin{aligned}\mathfrak{N}_{0,11}^{(0)}=&[(1-h_1)^2/(1+h_1)]\{(a_1^{(0)}+d^{(0)2}G_4)\\&\times[1-G_3(a_3^{(0)}+e^{(0)2}G_4)]\\&+G_3(c^{(0)}+d^{(0)}e^{(0)}G_4)^2\}\\=&0,\end{aligned} \tag{24}$$

and similarly for $M_{0,33}^{(0)}$

$$\begin{aligned}\mathfrak{N}_{0,33}^{(0)}=&[(1-h_3)^2/(1+h_3)]\{(a_3^{(0)}+e^{(0)2}G_4)\\&\times[1-G_1(a_1^{(0)}+d^{(0)2}G_4)]\\&+G_1(c^{(0)}+d^{(0)}e^{(0)}G_4)^2\}.\\=&0.\end{aligned} \tag{25}$$

These relations imply that

$$\begin{aligned}1-G_3(a_3^{(0)}+e^{(0)2}G_4)&=0,\\1-G_1(a_1^{(0)}+d^{(0)2}G_4)&=0,\\c^{(0)}+d^{(0)}e^{(0)}G_4&=0,\end{aligned} \tag{26}$$

and furthermore, once these are established, all other elements of $\mathfrak{N}_{0;ij}^{(0)}$ vanish at $s=s_g$ as well. The total number of independent $T=0$ parameters is reduced to two of which s_g is one and $a_3^{(0)}$ is arbitrarily taken as the other. In terms of these the other $T=0$ coupling parameters may be written:

$$\begin{aligned}a_1^{(0)}&=\frac{1}{P_1}\left\{1+(1-\eta_1)\left[\frac{\eta_4}{1-\eta_4}+\frac{\eta_3}{1-\eta_3}(1-a_3^{(0)}P_3)\right]\right\},\\c^{(0)2}&=\frac{1-a_1^{(0)}G_1}{G_1}\frac{1-a_3^{(0)}G_3}{G_3},\\d^{(0)2}&=\frac{1-a_1^{(0)}G_1}{G_1G_4},\\e^{(0)2}&=\frac{1-a_3^{(0)}G_3}{G_3G_4},\end{aligned} \tag{27}$$

in which the η_i's are defined by

$$\eta_i=P_i/G_i. \tag{28}$$

The search over the remaining three parameters in fitting (8) is carried out as in I. A grid of 21 (s,t) points in the region $0<s<4$, $0<t<s$ is taken. For each (s,t) $M_{11}^{(T)}(s,t)$ and $M_{11}^{(T)}(t,s)$ are computed for $T=0$ and for $T=2$ by summing a large number of the even partial waves. The quantity ψ_{02} is computed where

$$\psi_{02}=\frac{1}{\sqrt{2}N}\sum_{\substack{(s,t)\\ \text{grid}}}\frac{|\Delta^{(0)}+2\Delta^{(2)}|^2}{[\Delta^{(0)2}+4\Delta^{(2)2}]^{1/2}} \tag{29}$$

in which $N=21$. As explained in I the partial waves cannot be summed for $t>4$ since $t=4$ is a threshold above which the series diverges. For $s<0$ all the partial waves are complex but must be summed to give a real amplitude as long as $s>-t$; ignorance as to how this should be done confines the search to $s>0$.

Once this has been done the burden of fitting the remaining crossing relations rests on the $T=1$ channel. To carry this out a 5-parameter search is called for and no device is suggested here to improve on the techniques employed in I. To make a search as thorough as in the $T=0$ and $T=2$ cases would be far beyond the limits of reasonable computer time. It is, however, safe to conclude from the experience in I, that ψ's relevant to $T=1$ could be found which are as small as the minimum value of ψ_{02} found here.

One might contemplate fitting the crossing relations by this method in the other channels, e.g., application to the amplitude $M_{44}^{(0)}$. On the face of it this would seem to be simply a search through the two $T=0$ parameters in an effort to make $M_{44}^{(0)}$ a symmetric function of s and t. Such a search is impossible by this method for the same reason that the $M_{11}^{(T)}$ search is impossible for $s<0$. One is confronted with partial waves whose left-hand cuts begin to the right of $s=4$. The summed amplitude must be real in a region below $s=4$ but it is not clear how to perform the sum.

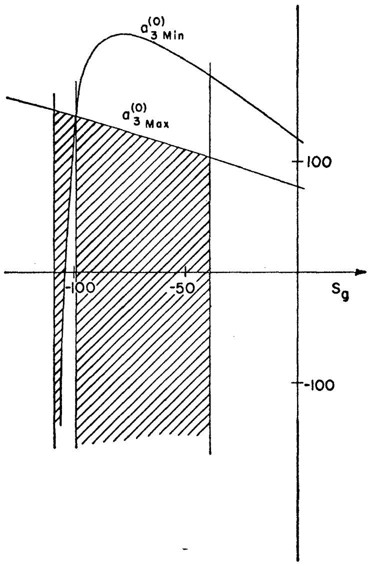

FIG. 1. Regions of allowed values of s_g and $a_3^{(0)}$.

3. RESULTS

Inspection of Eqs. (27) indicates that only certain regions of the parameters s_g and $a_3^{(0)}$ can be allowed; the requirement that $c^{(0)2}$, $d^{(0)2}$, and $e^{(0)2}$ all be non-negative implies that the regions for the search should be as shown in Fig. 1. The curves labeled $a_{3\ \max}{}^{(0)}$ and $a_{3\ \min}{}^{(0)}$ are defined by:

$$a_{3\ \max}{}^{(0)}=1/G_3\,,$$
$$a_{3\ \min}{}^{(0)}=(1-\eta_3\eta_4)/[G_3(1-\eta_4)]. \qquad (30)$$

The lines $\eta_1=1$, $\eta_3=1$, and $\eta_4=1$ form part of the boundaries. The best possible fit occurs for:

$$\begin{aligned} s_g&=-40\\ a_1^{(0)}&=45.4\\ a_3^{(0)}&=104\\ c^{(0)2}&=0.19\\ d^{(0)2}&=56.6\\ e^{(0)2}&=45.9\\ a_1^{(2)}&=-158 \end{aligned} \qquad (31)$$

and yields the result $\psi_{02}=0.262$. In Fig. 2 a map is given in which the result of computing $\Delta^{(0)}$ and $-2\Delta^{(2)}$ for each (s,t) point in the grid is plotted as a point in the $(\Delta^{(0)}, -2\Delta^{(2)})$ plane. The quantity ψ_{02} measures the average sine of the angle of departure from a perfect fit. The position of the ghost is determined to be in the neighborhood of the position conjectured by Chew and Frautschi.[3] As tacitly assumed in I, the $(K\bar{K})$ coupling to $(\pi\pi)$ is very small.

The phase shifts for $T=0$ are determined by computing:

$$\tan\delta_l{}^{(0)}=-\mathrm{Im}\mathfrak{D}_l{}^{(0)}/\mathrm{Re}\mathfrak{D}_l{}^{(0)} \qquad (32)$$

below the $(K\bar{K})$ threshold, and

$$\tan \mathrm{Re}\delta_l{}^{(0)}=-\mathrm{Im}(\mathfrak{N}_{l,11}{}^{(0)*}\mathfrak{D}_l{}^{(0)})/\mathrm{Re}(\mathfrak{N}_{l,11}{}^{(0)*}\mathfrak{D}_l{}^{(0)}) \qquad (33)$$

above the $(K\bar{K})$ threshold. For $T=2$ the formula is

$$\tan\delta_l{}^{(2)}=-\mathrm{Im}D_l{}^{(2)}/\mathrm{Re}D_l{}^{(2)}\,. \qquad (34)$$

The results of these calculations for s and d waves are given in Fig. 3.

The s-wave scattering length for $T=0$ is given by

$$a_s{}^{(0)-1}=\left[\left(\frac{p_1{}^2}{p_1{}^2+1}\right)^{1/2}\cot\delta_0{}^{(0)}\right]_{s=4}$$
$$=4\{[a_1{}^{(0)}+d^{(0)2}T_4+c^{(0)2}T_3(1-T_4/G_4)^2 \times[1-T_3(a_3{}^{(0)}+e^{(0)2}T_4)]^{-1}]^{-1}-T_1\}\,, \qquad (35)$$

and, for $T=2$, by

$$a_s{}^{(2)-1}=4(1/a_1{}^{(2)}-T_1)\,, \qquad (36)$$

in which the T_i's are defined by

$$T_i=H_i{}^{(l=0)}(4)\,. \qquad (37)$$

The results of the calculations are

$$a_s{}^{(0)}=-1.56\,,$$
$$a_s{}^{(2)}=-1.33\,, \qquad (38)$$

in which the $T=0$ figure is very nearly the same as that given by

$$a_s{}^{(0)-1}\approx 4(1/a_1{}^{(0)}-T_1)\,. \qquad (39)$$

It should be pointed out that the $T=0$ scattering length lies, for all parameters in the allowed regions of the search (Fig. 1), in the range -1.60 to -1.35, so that no choice of parameters leads to agreement with the number deduced from the ABC experiments[8]: $a_s{}^{(0)}=(2\pm1)\hbar/\mu c$, and with the number deduced from an analysis[9] of τ decay which includes p-wave π-π interaction: $a_s{}^{(0)}=1.96$.

The phase shift $\delta_0{}^{(0)}$ is shown in Fig. 3 to decrease through $-\frac{1}{2}\pi$ at $s=11.4$ (c.m. energy 473 MeV). This

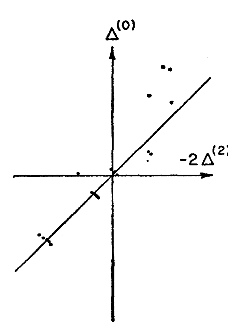

FIG. 2. Map of $\Delta^{(0)}$ versus $-2\Delta^{(2)}$.

[8] N. E. Booth and A. Abashian, Phys. Rev. **132**, 2314 (1963).
[9] M. A. Baqi Beg and P. C. DeCelles, Phys. Rev. Letters **8**, 46 (1962).

is not far from the position of the s-wave $T=0$ resonance conjectured by Brown and Singer[10] to give agreement to 3π decay modes of the η and K mesons. Experimental evidence [11] also exists for this resonance. The phenomenon at 473 MeV in this calculation, however, is not a resonance and therefore does not lend itself to comparison with Brown and Singer's analysis in spite of the fact that their width parameter enters squared throughout their calculations.

The phase shift $\delta_0^{(2)}$ has a real part far from $\frac{1}{2}\pi$ at the mass of the f^0.[12] This may not be discouraging in view of the fact that there exists the well-documented conjecture[13] that the f^0 might in fact be the neutral member of the ρ' or Buddha resonance.[14] Hopefully the ρ' would show up in the $T=1$ p wave but most likely not without inclusion of the $(\rho\rho)$ channel. That the ρ should occur in the $T=1$ p wave is safe to assume from experience in I.

4. OTHER TRIAL FUNCTIONS

The possibility that another choice of trial functions might be worth trying suggests itself. A choice having entirely different physical significance will be discussed in this section. A one-channel model of scalar meson scattering will be presented first to provide orientation for the $\pi\pi$ problem.

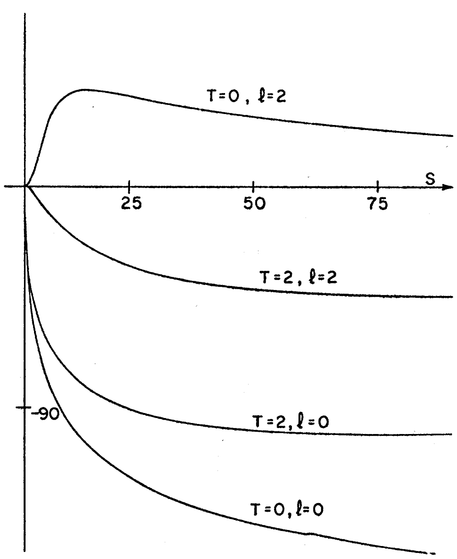

FIG. 3. s- and d-wave phase shifts.

[10] L. M. Brown and P. Singer, Phys. Rev. **133**, B812 (1964).

[11] N. P. Samios, A. H. Bachman, R. M. Lea, T. E. Kalogeropoulos, and D. Shepard, Phys. Rev. Letters **9**, 139 (1962).

[12] See, e.g., W. Selove, V. Hagopian, H. Brody, A. Baker, and E. Leboy, Phys. Rev. Letters **9**, 272 (1962).

[13] W. R. Frazer, S. H. Patil, and N. Xuong, Phys. Rev. Letters **12**, 178 (1964); see also, however, Y. Y. Lee, B. P. Roe, D. Sinclair, and J. C. Vander Velde, *ibid.* **12**, 342 (1964).

[14] M. Abolins, R. L. Lander, W. A. W. Mehlhop, N. Xuong, and P. M. Yager, Phys. Rev. Letters **11**, 381 (1963).

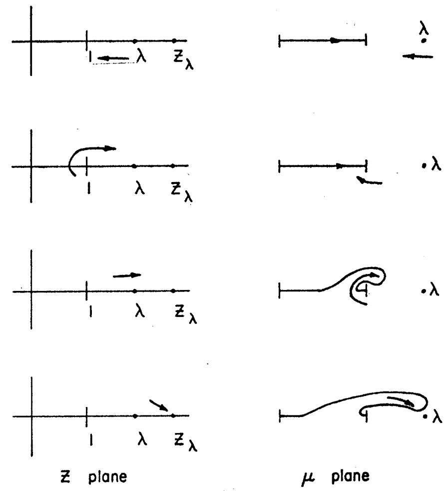

FIG. 4. Path of $\mu_\lambda(z)$ in μ plane corresponding to path of z in z plane.

A. Scalar Meson Scattering

A one-channel model for a variational approach to this problem might be defined by

$$M_l=N_l/D_l \tag{40}$$

with

$$N_l=aQ_l(\lambda)/2p^2 \tag{41}$$

in which

$$\lambda=\tfrac{1}{2}(h+1/h) \tag{42}$$

and in which h is the Khuri threshold factor appropriate to one meson exchange:

$$h=[(s-3)^{1/2}-1]/[(s-3)^{1/2}+1]. \tag{43}$$

The argument λ then becomes

$$\lambda=1+(1/2p^2) \tag{44}$$

so that the threshold behavior is given by

$$Q_l(\lambda)/(2p^2)\sim p^{2l}. \tag{45}$$

The primary consideration in the choice (41) can be seen by looking at first Born approximation

$$\begin{aligned}M_{\text{Born}(1)}&=\sum_{l\text{ even}}(2l+1)N_lP_l(z)\\&=\big(a/(4p^2)\big)\big(1/(\lambda-z)+1/(\lambda+z)\big)\\&=\tfrac{1}{2}a\big(1/(1-t)+1/(1-u)\big).\end{aligned} \tag{46}$$

That is, first Born approximation yields the one-meson exchange poles in the crossed channels. The denominator function is

$$D_l=1-d_l, \tag{47}$$

where

$$d_l=\frac{a}{2\pi}\int_4^\infty\frac{ds'}{s'-s}\frac{\rho'}{p'^2}Q_l(\lambda'), \tag{48}$$

where primes denote functions of s'. Second Born approximation is obtained from the second term in the expansion of D^{-1}:

$$D_l^{-1}=1+d_l, \quad (49)$$

and yields [neglecting the symmetrization performed in (46)]

$$M_{\text{Born}(2)}=\frac{a}{2p^2}\sum_l (2l+1)P_l(z)Q_l(\lambda)\frac{a}{2\pi}\times\int\frac{ds'}{s'-s}\frac{\rho'}{p'^2}Q_l(\lambda'). \quad (50)$$

The imaginary part is, for $s>4$ and $\lambda=1+1/(2p^2)$

$$\text{Im}M_{\text{Born}(2)}=\frac{a^2\rho}{4\pi p^4}\sum_l (2l+1)P_l(z)[Q_l(\lambda)]^2. \quad (51)$$

The sum can be written

$$\sum_l (2l+1)P_l(z)[Q_l(\lambda)]^2 =\tfrac{1}{2}\sum_l\int_{-1}^{1}\frac{d\mu}{\lambda-\mu}P_l(\mu)[Q_l(z\lambda-(z^2-1)^{1/2}(\lambda^2-1)^{1/2}) -2\sum_{m=1}^{l}(-)^m Q_l^m(\lambda)P_l^{-m}(z)]. \quad (52)$$

The first part of this expression can be summed for $1<z<\lambda$ to give

$$\tfrac{1}{2}\sum_l\int_{-1}^{1}\frac{d\mu}{\lambda-\mu}P_l(\mu)Q_l(\mu_\lambda(z))=\frac{1}{2}\int_{-1}^{1}\frac{d\mu}{\lambda-\mu}\frac{1}{\mu_\lambda(z)-\mu}, \quad (53)$$

where

$$\mu_\lambda(z)=z\lambda-(z^2-1)^{1/2}(\lambda^2-1)^{1/2}. \quad (54)$$

Now continue in z along the path indicated in Fig. 4; the pole at $\mu=\mu_\lambda(z)$ can be found to deform the contour of integration as shown in the figure, until the contour is pinched against the other pole at $\mu=\lambda$ as $z\to z_\lambda$. The point z_λ is given by

$$z_\lambda=2\lambda^2-1. \quad (55)$$

The position of this singularity in the t plane is given by

$$t(s)=-2p^2(1-z_\lambda) =4((s-3)/(s-4)); \quad (56)$$

this is the equation of the Mandelstam spectral curve for the scalar meson-box diagram. So, as in I, the choice of trial function (41) reflects desired analytic properties of the full amplitude.

The parameter a can now be varied to obtain a fit to the crossing relation

$$M(s,t)=M(t,s). \quad (57)$$

The summation of the partial waves is limited to the region $3<s<4$ and $3<t<4$. The search yields the result:

$$\psi_{\min}=0.464 \quad \text{with} \quad a=6.32.$$

One can now look at the s-wave amplitude with this value of a and ask whether unitarity and crossing have succeeded in dynamically generating the scalar meson as an s-wave bound state. The function D_0 vanishes at $s=-2.6$, corresponding to a ghost rather than a bound state. This calamity is presumably attributable to the fact that crossing can only be investigated in a small region which is too near the limit of convergence of the partial-wave series.

B. Pion-Pion Scattering

Analogously, the following model can be tried for the many-channel $\pi\pi$ problem, the channels being the same as specified in Sec. 2. The model is defined by taking $N=a$, the parameter matrix, in the expression

$$M_{l,ij}=q_i(ND^{-1})_{ij}q_j, \quad (58)$$

where

$$q_i^2=(1/(2p_i^2))\{Q_l(\lambda_i)+(2s/(\rho^2-4m_i^2))\times[Q_l(\lambda_i)-Q_l(\mu_i)]\}, \quad (59)$$

in which

$$\lambda_i=\tfrac{1}{2}(h_i+1/h_i), \quad (60)$$

$$\mu_i=\tfrac{1}{2}(k_i+1/k_i), \quad (61)$$

h_i is the Khuri threshold factor for ρ exchange between the particles in state i, and k_i is the Khuri factor for 2π exchange between the same particles. Both are incorporated to give a p-wave pole in first Born approximation at the mass of the ρ in the crossed channels.

If now one tries to impose the ghost constraints as in Sec. 2, one finds that there are no allowed regions of the two unconstrained parameters. That is, the ghost constraints cannot be imposed and the number of $T=0$ parameters cannot be reduced. A search with the full number of parameters is not feasible and so this model has been abandoned.

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Note added in proof. That the f^0 cannot be identified with the neutral Buddha has by now been established. See the second of Ref. 13 and the subsequent work of L. Sodickson *et al.* [Phys. Rev. Letters **12**, 485 (1964)], N. Gelfand *et al.* [*ibid.* **12**, 567 [1964)], and G. Benson *et al.* [*ibid.* **12**, 600 (1964)].